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ON THE STRUCTURE OF MORDELL-WEIL GROUPS OVER INFINITE NUMBER FIELDS

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1. INTRODUCTION

This is a written version of my talk under the same title at the Conference, except for the last section whose contents I did not mention in the talk. The first two sections are a résumé of my previous papers [8], [9] on the structure of the Mordell-Weil groups over a number field of infinite degree. In the last section, we discuss a generalization of our results from the view point of the gonality of curves contained in an abelian variety, and propose open questions.

I thank Professor Takao Komatsu for inviting me to this exciting conference and for financial support to participate in the conference.

Let A be a nonzero abelian variety defined over a number field K of finite degree over $\mathbb{Q}$. For an extension M over K, we denote the group of M-rational points by $A(M)$ and its torsion subgroup by $A(M)_{\mathrm{tors}}$. We call $A(M)$ is the Mordell-Weil group of A over M. It is well-known that $A(M)$ is a finitely generated abelian group for a finite algebraic extension M of K; then the Mordell-Weil rank means the rank of the torsion-free part of $A(M)$ as a free abelian group. On the other hand, for a number field of infinite degree its structure is not well-known. In this article we consider the Mordell-Weil group over infinite number fields; then the Mordell-Weil rank of A over an arbitrary M means $\dim_{\mathbb{Q}}(A(M)\otimes_{\mathbb{Z}}\mathbb{Q})$.

In [2], Frey and Jarden have asked whether the Mordell-Weil group of every nonzero abelian variety A defined over K has infinite Mordell-Weil rank over the maximal abelian extension K^{ab} of K. There are many results on this question. For elliptic curves E defined over $\mathbb{Q}$, Frey and Jarden proved the Mordell-Weil group $E(\mathbb{Q}^{\mathrm{ab}})$ has infinite rank. In [5], [15], [8], this is generalized to the Jacobian variety of a hyperelliptic curve defined over $\mathbb{Q}$. In fact, they showed the infiniteness of the Mordell-Weil rank for certain elementary abelian 2-extensions over $\mathbb{Q}$ and, in [8], we studied more precise structures of the Mordell-Weil groups in addition to the rank. Murabayashi [10] studied the Jacobians of superelliptic curves $y^p = f(x)$, where p is an arbitrary prime number, and showed the infiniteness of the rank for certain elementary abelian p-extensions over $\mathbb{Q}(\zeta_p)$. Rosen and Wong [12] proved the infiniteness of the rank for the Jacobian of any curve that can be realized over K as a cyclic geometrically irreducible cover of $\mathbb{P}^1$. Recently, Sairaiji and Yamauchi [13] proved the conjecture of Frey and Jarden for the Jacobians of non-singular projective curves defined over K under the assumption that the curves have infinitely many K^{ab}-rational points. Im and Larsen [4] proved the infiniteness of the Mordell-Weil rank for abelian varieties over any fields which have topologically cyclic absolute Galois groups and are not algebraic over finite fields.

2. Results

Our first result is the following:

Theorem 1. *Let C be a hyperelliptic curve of genus at least 1 defined over $\mathbb{Q}$ and let J be its Jacobian variety. Suppose that C has a $\mathbb{Q}$-rational point. Let K be a finite number field, and let $M = K(\sqrt{m} \mid m \in \mathbb{Z})$ be the field generated by all square roots of rational integers over K. Then the group $J(M)$ is the direct sum of a finite torsion group and a free $\mathbb{Z}$-module of infinite (countable) rank.*

This gives another proof of the results in [5], [15]. For a $\mathbb{Z}$-module X, that $\dim_{\mathbb{Q}}(X \otimes_{\mathbb{Z}} \mathbb{Q}) = \infty$ does not necessarily imply that X modulo torsion is a free $\mathbb{Z}$-module of infinite rank. Thus our statement above gives more precise information on the structure of $J(M)$ than those of [2], [5], [15]. It will be meaningful to study such precise structure of the Mordell-Weil groups as well as their ranks.

Two key ingredients in our proof are the following results of Ribet and Siegel.

Theorem 2. (Ribet, [11]) *Let $K(\zeta_\infty)$ be the field obtained by adjoining to K all roots of unity. Then for any abelian variety A over K, the group $A(K(\zeta_\infty))_{\mathrm{tors}}$ is finite.*

Since the field M in Theorem 1 is contained in $K(\zeta_\infty)$, the theorem of Ribet guarantees the finiteness of torsion subgroup $J(M)_{\mathrm{tors}}$.

Theorem 3. (Siegel, cf. [6]) *For an affine curve $C_0 \subset \mathbb{A}^n$ of genus at least 1 over K, the group of integer points $C_0(\mathcal{O}_K)$ is finite.*

For curves C of genus ≥ 2, we may appeal to Faltings' theorem [3] (= Mordell's conjecture) instead of Siegel's theorem.

Then we prepare a few algebraic lemmas, which are based on the finiteness of $J(M)_{\mathrm{tors}}$. Then these imply that the Mordell-Weil group with finite torsion group has free $\mathbb{Z}$-module structure modulo torsion:

Proposition 4. *Let A be an abelian variety over a number field K. Let M be a Galois extension of K such that $A(M)_{\mathrm{tors}}$ is finite. Then the group $A(M)/A(M)_{\mathrm{tors}}$ is a free $\mathbb{Z}$-module of at most countable rank.*

Remark. In my original talk, the extension M/K in Proposition 4 was not assumed Galois. After the talk, Professor Akio Tamagawa pointed out the Galois condition is necessary by providing a nice counterexample. The author thank him for this and some other useful comments.

By Proposition 4, it only remains to show that $J(M)$ is not finitely generated, and this can be proved by using Siegel's theorem.

In [8], in addition to Theorem 1, we exhibit some cases where, over certain larger fields, the Mordell-Weil groups modulo torsion are infinite-dimensional $\mathbb{Q}$-vector spaces.

Next, we generalized Theorem 1 to the Jacobians of superelliptic curves $y^n = f(x)$ defined over K (cf. [9]).

Theorem 5. *Let C be a smooth projective curve of genus ≥ 1 which is the smooth compactification of an affine plane curve defined by the equation $y^n = f(x)$ with coefficients in K, and let J be its Jacobian variety. Suppose that C has a K-rational point. Let $M = K(\sqrt[n]{m} \mid m \in \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of K. Then the Mordell-Weil group $J(M)$ is the direct sum of a finite torsion group and a free $\mathbb{Z}$-module of infinite rank.*

The key ingredient in the proof is the following variant of Theorem 2, which may be of some interest in its own right. We give here a proof of this Proposition which uses a different method from our original paper [9].

Proposition 6. *Let K be a number field and $K^{(n)}$ the composite field of all Galois extensions over K of degree $\leq n$. Then for any abelian variety A over K, the torsion group $A(K^{(n)})_{\mathrm{tors}}$ is finite.*

Proof. Let v be a finite place of K and w a place of $K^{(n)}$ lying above v. Let $K_w^{(n)}/K_v$ be the completion of $K^{(n)}/K$ at w. Then $K_w^{(n)}$ is the composite field of extensions over K_v of degree $\leq n$. By Serre's mass formula ([14]), the number of extensions of K_v with bounded degree is finite, and hence $K_w^{(n)}/K_v$ is a finite extension. Then Mattuck's theorem ([7], Thm. 7) implies the finiteness of torsion subgroup $A(K_w^{(n)})_{\mathrm{tors}}$. Hence we conclude that $A(K^{(n)})_{\mathrm{tors}}$ is finite. □

3. Open questions

Our results are of the cases where an abelian variety contains a hyperelliptic curve $y^2 = f(x)$ or a superelliptic curve $y^n = f(x)$. To generalize our results to a general abelian variety, it is useful to look at the gonality of curves embedded in the abelian variety. The gonality of a curve C means the lowest degree of a rational map from C to $\mathbb{P}^1$.

Along this line, Theorem 5 is converted to the following:

Let $K^{(n)}$ be the composite field of all Galois extensions over K of degree $\leq n$.

(a) If an abelian variety A over K contains an algebraic curve C which has a finite morphism $C \longrightarrow \mathbb{P}^1_K$ of degree $\leq n$, then

$$A(K^{(n)})/\mathrm{tors} \simeq \mathbb{Z}^{\oplus\infty}.$$

In fact, this follows by combining

(a′) If an algebraic curve C defined over K is n-gonal, then C has infinitely many $K^{(n)}$-rational points.

and

(a″) If an abelian variety A over K contains an algebraic curve C which has infintely many $K^{(n)}$-rational points, then the rank of $A(K^{(n)})$ is infinite.

On the other hand, Frey showed the following in [1].

(b) If an algebraic curve C defined over K has infinitely many $K^{(n)}$-rational points, then C is of at most $2n$-gonal.

This is close to the converse of (a′) and so it is natural to ask whether the converse of (a) holds or not:

(Q1) Let A be an abelian variety defined over K. Suppose the group $A(K^{(n)})$ has an infinite rank. Then does A contain a curve C of genus ≥ 2 and gonality $\leq n$?

In view of (b), we can ask a weaker question:

(Q1′) Let A be an abelian variety defined over K. Suppose the group $A(K^{(n)})$ has an infinite rank. Then does A contain a curve C of genus ≥ 2 and gonality $\leq 2n$?

By (b), this follows from:

(Q2) Suppose the group $A(K^{(n)})$ has an infinite rank. Then does A contain a curve C of genus ≥ 2 and having infinitely many $K^{(n)}$-rational points?

The question can be asked with an arbitrary extension of K (not only with $K^{(n)}$):

(Q3) Let M be an algebraic extension of K. Suppose the group $A(M)$ has an infinite rank. Then does A contain a curve C of genus ≥ 2 and having infinitely many M-rational points?

References

[1] G. Frey, *Curves with infinitely many points of fixed degree*, Israel J. Math. **85** (1994), 79–83.

[2] G. Frey and M. Jarden, *Approximation theory and the rank of abelian varieties over large algebraic fields*, Proc. London Math. Soc. **28** (1974), 112–128.

[3] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349–366.
G. Faltings, *Erratum: "Finiteness theorems for abelian varieties over number fields"*, Invent. Math. **75** (1984), no. 2, 381.

[4] B. Im and M. Larsen, *Abelian varieties over cyclic fields*, to appear Amer. J. Math.

[5] H. Imai, *On the rational points of some Jacobian varieties over large algebraic number fields*, Kodai Math. J. **3** (1980), 56–58.

[6] S. Lang, *Fundamentals of Diophantine Geometry*, Springer-Verlag, 1983.

[7] A. Mattuck, *Abelian varieties over p-adic ground fields*, Ann. of Math. **62** (1955), 92-119.

[8] H. Moon, *On the Mordell-Weil groups of Jacobians of hyperelliptic curves over certain elementary abelian 2-extensions*, to appear in Kyungpook Math. J.

[9] ———, *On the structure of the Mordell-Weil groups of the Jacobians of curves defined by* $y^n = f(x)$, preprint.

[10] N. Murabayashi, *Mordell-Weil rank of the Jacobians of curves defined by* $y^p = f(x)$, Acta Arith. **64** (1993), 297–302.

[11] K. Ribet, *Torsion points of abelian varieties in cyclotomic extensions*, appendix to N. Kats and S. Lang, *Finiteness theorems in geometric classfield theory*, Enseign. Math. (2) **27**(1981), no. 3-4, 285–319.

[12] M. Rosen and S. Wong, *The rank of abelian varieties over infinite Galois extensions*, J. Number Theory **92** (2002), 182–196.

[13] F. Sairaiji and T. Yamauchi, *The ranks of Jacobian varieties over the maximal abelian extensions of algebraic number fields: Toward Frey-Jarden's conjecture*, preprint.

[14] J.-P. Serre, *Une "formule de masse" pour les extensions totalement ramifiées de degré donné d'un corps local*, C. R. Acad. Sc. Série A **286** (1978), 1031–1036.

[15] J. Top, *A remark on the rank of Jacobians of hyperelliptic curves over $\mathbb{Q}$ over certain elementary Abelian 2-extension*, Tohoku Math. J. **40** (1988), 613–616.

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